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Relating Classical Mechanics to Quantum Mechanics

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1. Introduction

At the atomic and subatomic levels, the world is governed by quantum mechanics, whereas the macroscopic world we directly perceive follows classical mechanics. Systems composed of vast numbers of atoms appear to obey laws different from those of the quantum domain. Historically, classical mechanics preceded quantum mechanics, but its limitations became evident in the late 19th century when experiments contradicted its predictions. This led to the development of quantum mechanics in the early 20th century, which successfully explained these anomalies and introduced several radical conceptual shifts.

Over time, physicists came to regard classical mechanics as a limiting case of quantum mechanics—for large systems or high energies. However, this view was often based on specific examples rather than a fully established logical derivation.

This stand raises a fundamental question: Can the formal structure of classical mechanics be rigorously derived from quantum mechanics? Attempts using the Schrödinger or Heisenberg formulations did not fully succeed in deriving Newton's laws or the Lagrangian formalism. Ehrenfest showed that expectation values of quantum observables approximately follow Newton's laws, but this did not constitute a complete derivation.

A significant advance came with Feynman's path integral formulation, inspired by Dirac's insights into the conceptual foundations of quantum theory. The path integral approach not only clarified conceptual issues in earlier formulations but also provided a logically consistent framework showing how classical mechanics emerges from quantum mechanics.

Before examining this derivation, it is useful to briefly contrast the fundamental features of classical and quantum laws.

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2. Classical and Quantum Laws in Brief

In classical mechanics, a system is described by precisely defined position and momentum at every instant in a chosen reference frame. Given the forces acting on it, future motion can be predicted with certainty using Newton's laws or, more generally, the Lagrangian formalism. All other observable properties follow from position and momentum (ref.1). Classical dynamics is therefore **deterministic**. (We are only considering the domain of regular behaviour here).

In contrast, quantum mechanics does not assign definite position and momentum simultaneously. Instead, a system is described by a state vector in a finite or an infinite dimensional Hilbert space, commonly represented by a wave function or a probability amplitude. This wave function evolves according to the Schrödinger equation and yields only probabilistic predictions for measurement outcomes, as interpreted in the Copenhagen framework (ref.2 and ref.6). The probabilistic nature of results and the complementary wave–particle character of matter distinguish quantum theory fundamentally from classical mechanics.

It is this contrast that prompts and triggers any one to demand a convincing justification regarding how the path integral formalism that describes the quantum behaviour of systems can harbour in itself a formalism that is applicable to classical systems. More explicitly, how does the formalism that accounts for the probabilistic predictions of quantum mechanics can lead to the precisely predictable results of the classical mechanics. The present article is aimed at illustrating the passage that bridges the two seemingly different approaches through the double-slit experiment with electrons - an experiment which was initially conceptualised to show the dual behaviour of electrons and was experimentally demonstrated decades later after the birth of quantum mechanics (ref.4).

3. Path Integral formalism

Consider a quantum system evolving under a time-independent Hamiltonian, $H = \frac{p\hat{}2}{2m} + V$. From the Schrödinger equation, the time evolution between t_1 and t_2 is,

$$|\psi(t_f)\rangle = U(t_f, t_i)|\psi(t_i)\rangle = e^{\frac{-i(t_f - t_i)H}{\hbar}} |\psi(t_i) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1)$$

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Suppose that at time t_i the particle is localized at position x_i, so that its state is $|\psi(t_i)\rangle = |x_i\rangle$. We want the probability amplitude that at time t_f, the particle is found at position x_f. This transition amplitude is (ref.8)

$$\langle x_f|\psi(t_f)\rangle = \langle x_f|e^{\frac{-i(t_f-t_i)H}{\hbar}}|x_i\rangle \qquad \ldots\ldots (3.2)$$

In the path-integral formulation (ref.7), this transition amplitude is interpreted in a striking way:

- The particle does **not** travel along a single path.
- Instead, it explores **all possible forward paths** connecting x_i and x_f.
- Each path contributes a complex probability amplitude.
- The total amplitude is obtained by **adding contributions from all permissible paths**.

Thus, the transition from the initial state to the final state is viewed as a coherent sum over every admissible path between them. To see how different paths arise and contribute, we consider the classic example of the double-slit experiment with electrons.

4. The Double-slit Experiment with Electrons

It is well known that Thomas Young in 1802, performed an experiment with light passing through a double-slit (ref 3). He clearly demonstrated the wave nature of light rebutting the corpuscular nature of light that was doing the rounds at that time.

Here, monochromatic electrons emanating from a point source are incident on a double-slit (of suitable size) as shown in figure 1. It may be noted that in the improved version of this experiment, electrons were sent one after the other, the second one made to incident on the double slit only after the first one reached the photographic plate kept sufficiently far away from the double-slit. This care was taken to avoid the possibility of mutual interaction between the electrons during the transit, which would possibly affect their behaviour in the experiment.

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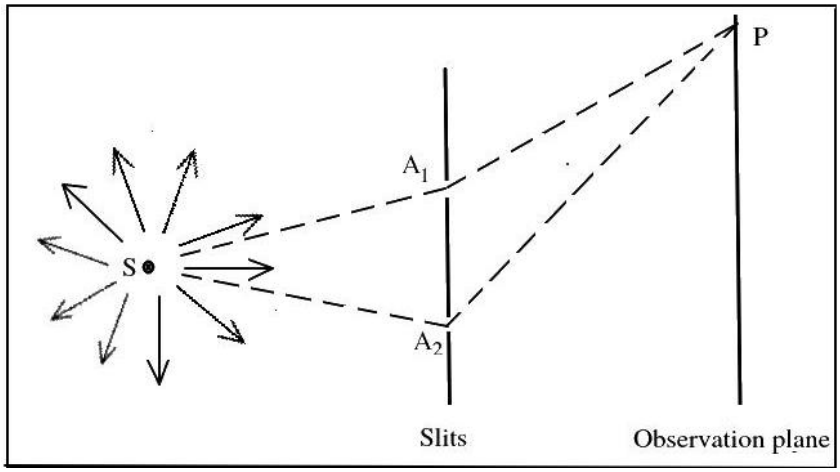

Fig. 1: Electrons incident on a double slit

It is found from such experiments (ref. 4) that, when electrons are sent one by one, the intensity of electrons at a general point like P shows an oscillatory (or periodically varying) pattern varying from zero to a maximum positive value depending on the position of P. It must be noted here that each electron appeared to fall randomly on the screen, but over a length of time, the cumulative effect developed into an interference pattern. Though the electrons were identical in all respects, yet, somehow, they ended up showing an interference pattern exactly like that of light. Feynman had described this experiment on conceptual grounds earlier to highlight the quantum nature of electrons. The interference pattern on the screen may be successfully explained using quantum ideas such as electrons also have dual behaviour and in this experiment, they register their arrival at some point like a particle on the photographic plate, but in transit, they pass through both slits with a wavelength given by the de Broglie relation, $\lambda = h/p$ and energy $E = h\nu$, where h is the well –known Planck's constant. Though the experimental demonstration of this was done much later, Feynman's path integral formalism provides an elegant explanation of this quantum behaviour of electrons, it also provides the rationale which forms the basis for classical formalism applicable to macroscopic objects. It also demonstrates that electrons travel along forward going paths only.

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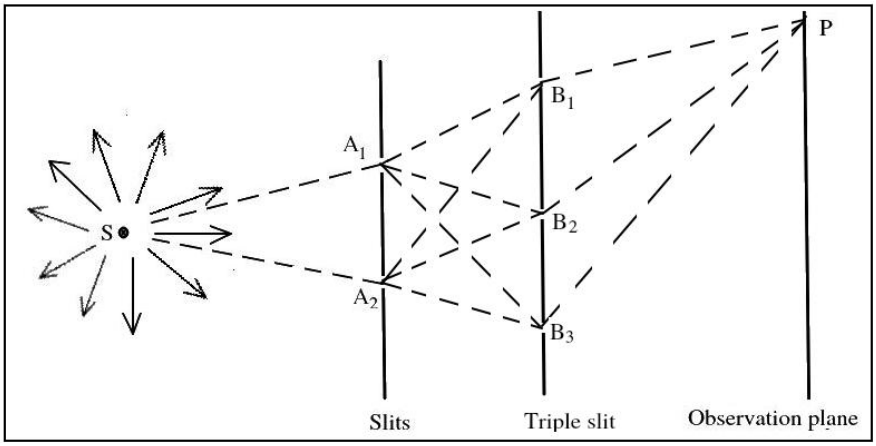

Fig (2): Triple slit between the double slit and the screen

To illustrate the way the path integral formalism is applied here, let us continue with the above experiment and place another screen with three slits between the double slit and the screen P as shown in figure 2.

Following the same logic as in the double bit interference case, one can say that the electrons reach a general point P via all possible forward paths from A_1 and A_2, after passing through all the slits B_1, B_2, and B_3 – finally ending up on the screen producing an interference pattern. Let us generalise to the case of the intermediate screen with almost infinite number of slits placed at infinitesimally close positions between the source and the point of observation like P.

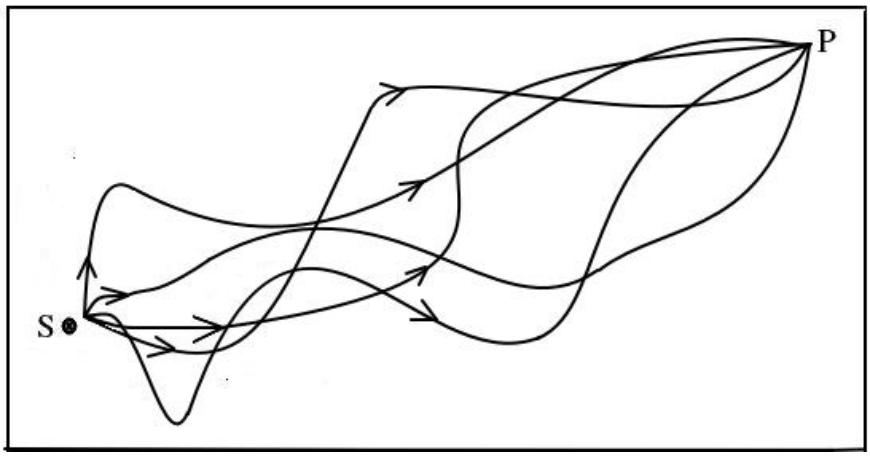

Fig (3): Infinite number of paths between S and P

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In this situation, one can see that there can be infinitely many paths, as shown in the figure (3), that are available for the electron and every such path connecting S and P is equally probable. There are of course some restrictions on the nature of paths as mentioned in the box below.

Quantum mechanically allowed paths

In a situation as in the double-slit experiment, quantum mechanics allows continuous paths that move forward from the start to the end. They may not be differentiable but must be continuous. Paths with multiple values for the same instant of time are not allowed. Figures (4) and (5) below show, respectively, some examples of possible allowed and disallowed paths. This is similar to Huygens paths for optical rays(ref.5).

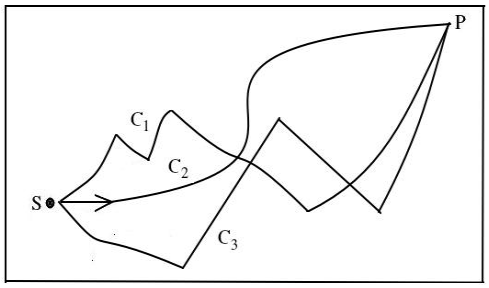

Fig (4): Allowed paths between S and P

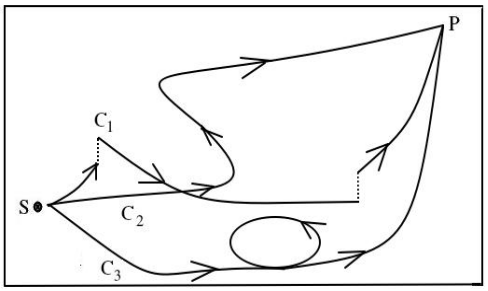

Fig (5): Disallowed paths

Feynman proceeded by associating a probability amplitude K (x_f, x_i) to each such path available for the electron in state $|x_i\rangle$ at time t_i to be found at $|x_f\rangle$ at time t_f as given by equation (3.2). i.e.

$$K(x_f, x_1) = \langle x_f|\psi(t_2)\rangle = \langle x_f|e^{\frac{-i(t_f-t_i)H}{\hbar}}|x_i\rangle \quad \text{..... (4.1)}$$

To evaluate this amplitude, one can break up the path, as shown in Figure (6), the finite time interval ($t_f - t_i$) into a very large number N of small equal intervals of width ϵ so that $t_f - t_i = N\epsilon$. We can now write the above expression as

$$K(x_f, t_f; x_i, t_i) = \langle x_f|\psi(t_f)\rangle = \langle x_f|\left(e^{\frac{-iH}{\epsilon}}\right)^N|x_i\rangle$$

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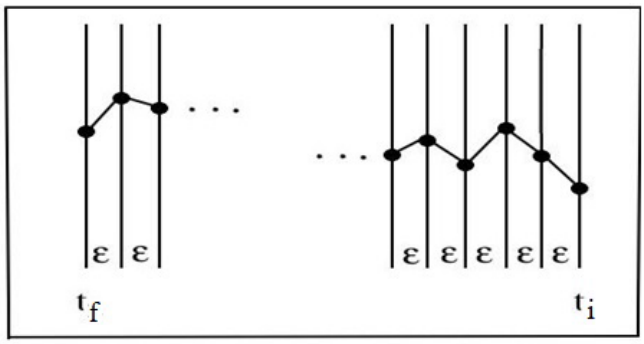

Fig. 6: Evolutionary paths

The sum of amplitudes over all paths is now done by introducing the intermediate position eigenstates of position and integrating over all of them (the associative rule). We then have:

$$K(x_f, t_f; x_i, t_i) = \langle x_f | \left(e^{\frac{-iH}{\epsilon}} \right)^N |x_i\rangle =$$

$$\int..\int dx_1\, dx_2\,..\langle x_f | e^{\frac{-iH\epsilon}{\hbar}} |x_N\rangle..\langle x_2| e^{\frac{-iH\epsilon}{\hbar}} |x_1\rangle \langle x_1| e^{\frac{-iH\epsilon}{\hbar}} |x_i\rangle \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.2)$$

To evaluate one of the matrix elements $\langle x_{j+1}| e^{\frac{-iH\epsilon}{\hbar}} |x_j\rangle$, we use $H = \frac{p^2}{2m}$, which is true for free electrons and the resolution of identity: $\int dp_i \langle p_i | p_i\rangle = 1$ for momentum eigenstates as well as the well- known Gaussian integral formula to get

$$\langle x_{j+1}| e^{\frac{-iH\epsilon}{\hbar}} |x_j\rangle = \int dp\ \langle x_{j+1}| e^{\frac{-iH\epsilon}{\hbar}} |p\rangle\langle p|x_j\rangle$$

$$= \frac{1}{2\pi\hbar}\int e^{\frac{ip\left(x_{j+1}-x_j\right)}{\hbar}} e^{-i\frac{p^2}{2m\hbar}\epsilon} dp = \sqrt{\frac{m}{2\pi i\hbar\epsilon}}\ e^{\frac{im}{2\hbar}\left(\frac{x_{j+1}-x_j}{\epsilon}\right)^2\epsilon} \ldots\ldots\ldots\ldots\ldots \quad (4.3)$$

Inserting equation (4.3) in to equation (4.2) and letting $N \to \infty$, we get,

$$K(x_f, t_f; x_i, t_i) = \lim_{N\to\infty} \int..\int dx_1\, dx_2\,..\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{N/2}\ e^{\frac{i}{2\hbar}\sum_{j=1}^{N}\frac{\epsilon m}{2\hbar}\left(\frac{x_{j+1}-x_j}{\epsilon}\right)^2}$$

Denoting $\Delta t = \epsilon$, $\Delta x = x_{j+1} - x_j$, the above can also be expressed as

$$K(x_f, t_f; x_i, t_i) = \lim_{N\to\infty} \int..\int dx_1\, dx_2\,..\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{N/2}\ e^{\frac{i}{\hbar}\int dt\,\frac{mv\hat{}2}{2\hbar}} \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.4)$$

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Identifying the kinetic energy $\frac{mv\hat{}2}{2}$ in the exponent as the Lagrangian L, we thus have,

$$K(x_f, t_f; x_i, t_i) = \lim_{N\to\infty} \int..\int \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{N/2} dx_1\, dx_2\,.\, e^{\frac{i}{\hbar}\int L\, dt} \quad \ldots\ldots\ldots\ldots\ldots \quad (4.5)$$

This expression can be generalised to include the potential V(x) also (ref.8) and using the definition of classical action functional: $S=\int L\, dt$, we will have,

$$K(x_f, t_f; x_i, t_i) = \int D[x(t)]\; e^{\frac{iS[x(t)]}{\hbar}} \quad \ldots\ldots\ldots\ldots\ldots \quad (4.5)$$

Where $\int D\,[x(t)]$ denotes a sum over all allowed paths between x_i and x_f, ignoring mathematical complexities that are to be addressed in rigorous treatments.

5. Emergence of Classical Mechanics

When the action S is large compared to ħ, as it happens for classical systems, the exponential oscillates extremely rapidly. Hence, destructive interference cancels contributions from most paths. Only paths for which the action integral is stationary for small variations around a path i.e., δS = 0 survive. This condition yields the Euler–Lagrange equation of classical mechanics:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5.1)$$

6. Summary and Concluding Remarks

The path integral formulation provides a unified and conceptually transparent framework for understanding both quantum and classical mechanics. In this formulation, the probability of amplitude for a system to evolve from an initial state to a final state is expressed as a sum over all possible continuous paths connecting the two configurations. Each path contributes a complex amplitude weighted by the phase factor exp (iS[x (t)]/ $\hbar$), where S [x (t)] is the action functional associated with that path.

Careful examination of the results of the double slit experiment shows that only continuous and forward oriented paths are allowed but not any and wildly varying paths, as sometimes imagined in popular presentations. A crucial insight emerges when we examine the role of the ratio S/$\hbar$. For microscopic systems, action S is comparable to Planck's constant $\hbar$, and therefore the phase varies moderately between neighbouring paths. As a result, a wide range of paths

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contributes significantly to the total amplitude, and quantum interference effects become observable.

In contrast, for macroscopic systems, the action is enormously large compared to $\hbar$. In this regime, the phase factor oscillates extremely rapidly for small variations of the path. Contributions from neighbouring non-stationary paths cancel one another through destructive interference. Only those paths for which the action is stationary—meaning that small variations do not change it to first order—survive constructively. Mathematically, this condition, as mentioned earlier, is expressed as $\delta S=0$, which is precisely the principle of stationary action.

Thus, classical mechanics emerges naturally as the stationary-phase approximation of the quantum mechanical path integral. In this sense, classical mechanics is structurally embedded within quantum mechanics! Historically, the action principle introduced by Euler, Lagrange and Hamilton appeared as an independent and somewhat mysterious postulate of classical mechanics. In the path integral formulation, however, the action functional arises naturally from the short-time quantum propagator derived from the Schrödinger equation. The classical Lagrangian structure is therefore seen not as an isolated assumption but as a natural consequence of quantum time evolution.

It is also important to emphasize that the transition between quantum and classical behaviour is not sharply defined. Since Planck's constant is finite, the distinction between the two domains depends on the relative magnitude of the action compared to $\hbar$. Practical considerations, including the scale of the system and the nature of measurement, influence whether quantum interference effects are observable or effectively suppressed. Our measurement apparatus and environmental interactions play a significant role in determining where the classical description becomes adequate.

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